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Investigation of Coupled Harmonic Oscillations with a V-Scope

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The study of coupled mechanical harmonic oscillations in the undergraduate physics laboratory is enhanced by the use of a V-Scope which enables students to achieve a high standard of precision in the acquisition and analysis of computerized data. The V-Scope, a new tool for making motion measurements, is made in Israel by Litek[1] and marketed in the United States by PASCO[2]. The device consists of three major components: a microcomputer, three towers, and a set of buttons. In addition, a personal computer is required. The microcomputer controls the V-Scope and interfaces with the personal computer. The three towers emit simultaneously an infrared signal that activates the buttons, and then the towers pick up a return ultrasonic signal from the buttons. The buttons are attached to the moving parts of a physics experiment. The time of flight of the ultrasonic signal to each tower is measured, and the microcomputer computes the position of each of the buttons. A complete assessment of the V-Scope may be seen in Daw (1991).

In this paper, experiments with equal-length and different-length coupled pendulums were carried out measuring the frequency of energy exchange between them. Both pendulum frequency and energy exchange frequency were measured as functions of the coupling height and of the coupling spring constant. The experiments described in this paper would be most appropriate for an introductory mechanics laboratory course and had just been introduced in the physics laboratory matriculation examination at the end of the 12th grade. Both teachers' and high-school and university students' reactions were very positive. They emphasized primarily the ease of use and the possibility to concentrate on the physical phenomena instead of on the time-consuming and tedious work of data collection.

INTRODUCTION

According to Mestre (1991), the most prevalent instructional practice in school is the so-called transmission model of instruction. In this model, students are exposed to content through lectures and demonstrations, and are expected to absorb the transmitted knowledge in ready-to-use form. He claims that

> The transmission model is used largely by default rather than choice, both because it is the instructional method by which we were taught and because it may be the only instructional method we know. Not only does it have little theoretical justification, but there is mounting evidence that it is not the most efficient method of instruction. (p. 56)

Results from research in cognitive science and education verify the importance of basing development of scientific concepts and skills on concrete experience (Arons, 1983; Rosenquist & McDermott, 1987). Arons (1983) claims that "it is essential to give students a chance to follow and absorb the development of a small number of major scientific ideas" (p. 97) at a pace that makes their knowledge operative (understanding the source of knowledge) rather than declarative (knowing facts).

Many researchers (Laws, 1991; Morse, 1993; Thornton & Sokoloff, 1990) have developed microcomputer-based laboratory (MBL) tools and curricula that can help students make connections between the physical world and the underlying principles that constitute scientific knowledge. As Thornton and Sokoloff (1990) claim, "these materials, which are intended for use in introductory courses in high school and college, provide a convenient and effective means for collecting and displaying physical data in a form that students can remember, manipulate, and think about" (p. 858). The effectiveness of the MBL tools and curriculum in teaching kinematics (Thornton & Sokoloff, 1990) and dynamics (Morse, 1993) has encouraged the development of tools and curriculum to continue to teach topics in classical mechanics (Eckstein & Fekete, 1991).

It is relatively easy to carry out motion experiments involving most of the topics included in a classical mechanics curriculum. Some of these phenomena, like coupled harmonic oscillations, are too fast to follow in order to acquire and analyze data. Though real processes can be generated easily, their further investigation relies mainly on films or computer simulations. In this paper, we describe the use of a the MBL tool called V-Scope (Ronen & Lipman, 1991) as a means of acquiring and analyzing data to study coupled

harmonic oscillations. The introduction of V-Scope to the physics laboratory allows students to investigate this important subject in ways that have not been available up to now. A graph of the motion of the different oscillators is produced on the computer monitor in real time, allowing students to observe the mechanical oscillations and their graphical representation simultaneously. The usual span of a run is of the order of 30-60 seconds, slow enough to observe what is happening and fast enough so that it does not become tedious.

We shall describe the following laboratory experiments which would be most appropriate for a laboratory course based on Vol. 1 of the Berkeley Physics Course (Kittel, Knight, & Ruderman, 1962) or other similar texts:

1. Measurement of the frequency of energy exchange between two equal-length pendulums coupled by a rubber-band.
2. Measurement of the frequency of energy exchange between two different-length pendulums coupled by a rubber-band.
3. Investigation of the functional dependence of the pendulum frequency and of the energy exchange frequency between two pendulums coupled by a rubber-band on the height of the coupling point.
4. Investigation of the functional dependence of the pendulum frequency and of the energy exchange frequency between two pendulums coupled by a spring on the spring constant k.

THE V-SCOPE

The most commonly used microcomputer-based systems for studying motion in the undergraduate physics laboratory (Crummet & Western, 1987; Eckstein & Fekete, 1991) are sonic rangers (Cline & Risley, 1988). They measure the motion of one or more bodies in one dimension, producing experimental data for students to analyze by their own means. The V-Scope (Instruction Manual V-Scope System, 1990), recently developed in Israel, tracks and analyzes three-dimensional multibody motion. It continuously measures and records the time functions of the position vectors **R**(t) of each moving body. The other physical magnitudes, such as velocity and acceleration, are obtained from **R**(t) by mathematical and graphical processing. The V-Scope parts which execute the tracking process (see Figure 1) are (IMVS, 1990):

- A button attached to each movable body; each button incorporates an infrared receiver and a synchronized ultrasonic transmitter. The button is essentially the body which is tracked by the V-Scope. Buttons are identified by different colors.

- Fixed towers which communicate with the buttons. Each tower incorporates an infrared transmitter and an ultrasonic receiver.
- The V-Scope microcomputer, which controls the operation of the towers, initiates outgoing signals, processes incoming signals, and calculates the spatial position of the buttons in cartesian coordinates, then transfers them to the personal computer.

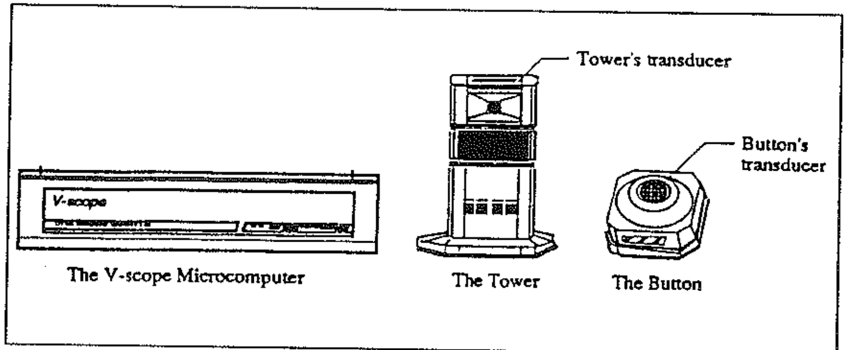

Figure 1. V-Scope parts

Results are instantly displayed on the computer monitor and the procedure is performed at high speed, repeated with a rate up to 100 Hz. The resolution of the position measurement is about 0.5 mm within an effective measuring space of 5 * 5 * 5 m.

EXPERIMENTAL SETUP

The experimental setup is shown in Figure 2. Two balls (151.5 gr mass, 30 mm radius) are hung on flexible strings from hooks on the ceiling. The strings, coupled by a rubber-band, are 1.38 m long so that the pendulums are hanging 50 cm from the experiment table. The buttons (yellow and green) are attached to the bottom of the pendulums at their center. The towers, "looking" upwards, are arranged in a three-dimensional horizontal configuration. The motion of both pendulums, measured simultaneously, is aligned with the *y*-axes.

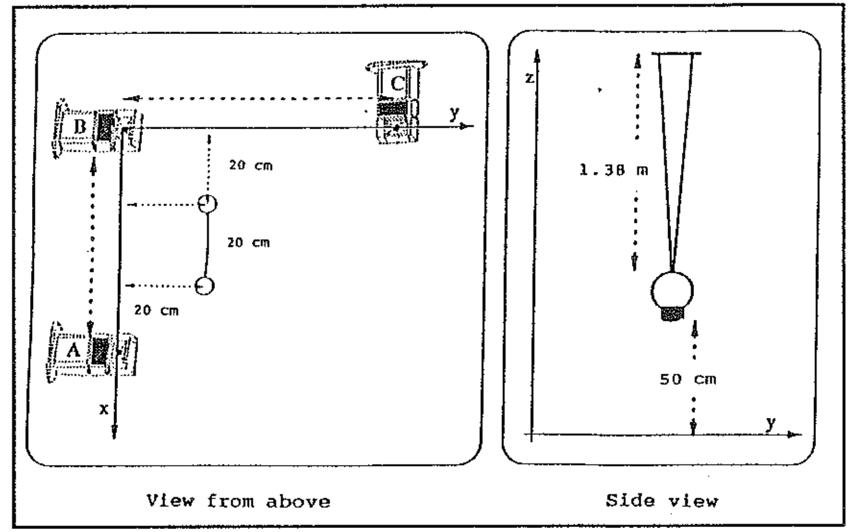

Figure 2. Setup for measurement of coupled oscillations, pendulums moving in y-direction

THE EQUATION OF MOTION AND ITS SOLUTION

To discuss the problem dynamically, we must set up the equation of motion of each pendulum. A simple pendulum is defined as a particle of mass m suspended by a string of length L and of negligible mass. If the particle is pulled aside so that the string makes an angle with the vertical, and then released, the pendulum will oscillate. The particle moves in an arc of a circle of radius L. The forces acting on the particle are its weight (mg) and the tension (T) along the string. The tangential component of the resultant force is

$$F_T = mg*\sin\theta$$

The equation for the tangential motion is $F_T = ma_T$, where a_T is the tangential acceleration. Since the particle moves along a circle of radius L, $a_T = L*d^2\theta/dt^2$. The equation for the tangential motion is thus (Alonso & Finn, 1969):

$$mL*d^2\theta/dt^2 = -mg*\sin\theta \quad \text{or} \quad d^2\theta/dt^2 + (g/L)\sin\theta = 0 \qquad (1)$$

In the limit of small deflection angles, $\sin\theta = \theta$, and Equation (1) becomes

$$(d^2\theta/dt^2) + (g/L)\theta = 0 \qquad (2)$$

This is the equation of motion of a harmonic oscillator of angular frequency

$$\omega = (g/L)^{1/2} \qquad (3)$$

Thus, we may conclude that, within such an approximation, the angular motion of the pendulum is simple harmonic and the angle can be expressed in the form $\theta = \theta_0 * \sin(\omega t + \alpha)$, where α is an arbitrary initial phase.

In this study, we consider the case of two pendulums of lengths L_1 and L_2, attached by some coupling device, displaced by angles θ_1 and θ_2 from their equilibrium positions.

Since the coupling device exerts an additional force which can be seen as proportional to the displacement of each of the pendulums, the coupling constant being named δ, the equation of motion for each pendulum may be written as (Alonso & Finn, 1969):

$$(d^2\theta_1/dt^2) + \{(g + \delta)/L_1\}\theta_1 = (\delta/L_1)\theta_2$$

and (4)

$$(d^2\theta_2/dt^2) + \{(g + \delta)/L_2\}\theta_2 = (\delta/L_2)\theta_1$$

The left-hand sides of these equations are very similar to Equation (2), except for the terms δ/L_1 or δ/L_2 added to the frequencies of the oscillators. Another difference is that instead of zero on the right-hand side, we have a term referring to the other oscillator, the coupling term. Instead of attempting to obtain the general solution of Equation (4), we shall limit ourselves to the special case of two identical oscillators, so that $L_1 = L_2$. It can be proved that in this case the general motion of the two coupled pendulums may be considered as the superposition of two "normal modes" (Alonso & Finn, 1969, p. 368) of oscillation. In one of the normal modes, the two pendulums move in phase with equal amplitudes.

That is,

$$\theta_1 = A_1 * \sin(\omega_1 t + \alpha_1), \quad \theta_2 = A_1 * \sin(\omega_1 t + \alpha_1) \qquad (4)$$

where

$$\omega_1 = (g/L)^{1/2} \qquad (5)$$

That is, the frequency of the coupled oscillators is the same frequency of oscillation which each pendulum would have if there were no coupling. This is easily understood because, since the two pendulums have the same amplitude and are in phase, the coupling device does not exert any force on the pendulums, which move as if they were uncoupled.

In the second normal mode, the two pendulums move in opposition with equal amplitude:

$$\theta_1 = A_2\sin(\omega_2 t + \alpha_2), \quad \theta_2 = -A_2\sin(\omega_2 t + \alpha_2) \qquad (6)$$

where

$$\omega_2 = \{(g + 2\delta)/L_1\}^{1/2} \qquad (7)$$

That is, the frequency is higher than the frequency without coupling. This is also understood because now the coupling device is stretched and compressed, exerting an additional force which increases the oscillating frequency.

The general solution (Alonso & Finn, 1969) involves a linear combination of the normal modes of oscillation. In the special case of equal amplitudes and assuming that the initial phases are zero ($\alpha_1 = \alpha_2 = 0$), we get:

$$\theta_1 = [2A_1\cos\tfrac{1}{2}(\omega_1 - \omega_2)t]\sin\tfrac{1}{2}(\omega_1 + \omega_2)t$$

and (8)

$$\theta_2 = [2A_1\sin\tfrac{1}{2}(\omega_1 - \omega_2)t]\cos\tfrac{1}{2}(\omega_1 + \omega_2)t$$

Comparing these expressions we see that the two modulating amplitudes have a phase difference of $\pi/2$, or a quarter of the modulating period. Because of the phase difference between the two modulating amplitudes, there is a continuous exchange of energy between the two pendulums, as will be measured in the present study.

FREE OSCILLATIONS

In a preliminary experiment, the original frequencies of the two uncoupled pendulums was measured (see Figure 3).

The equations for the horizontal displacement of each of the pendulums, fitted by the SAS Nlin procedure (SAS/STAT User's Guide, 1990), and neglecting a slight damping due to friction, are

$$y(t) = A * \cos(\omega t)$$

The fitted parameters are shown in Table 1.

We can see that we have two slightly damped harmonic oscillations of the same frequency within the measurement error limits. The value of the frequency calculated from Equation (3) is 2.635 ± .005 rad/s, which is within the error limits of the measured value for the "yellow" pendulum and at the limit of the measured value for the "green" pendulum.

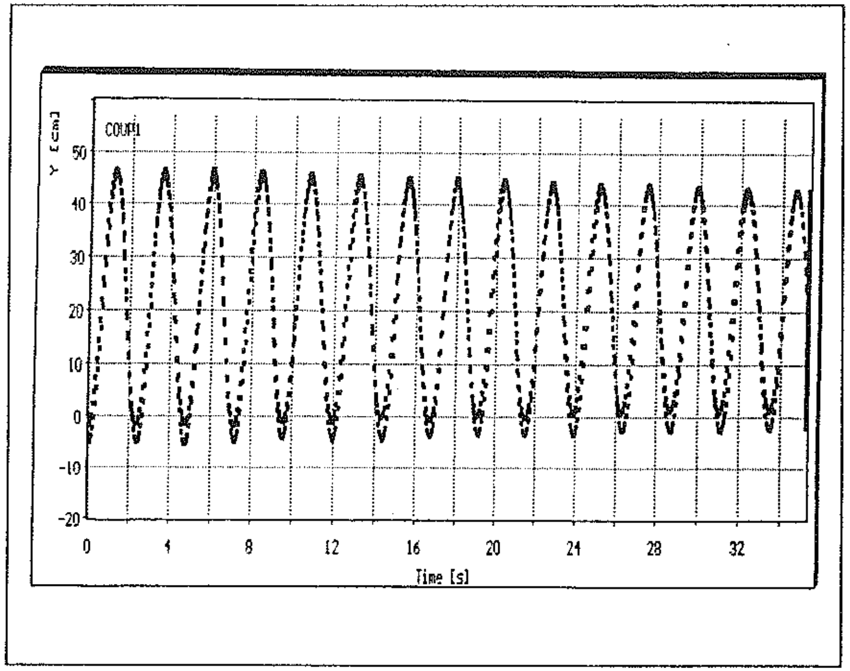

Figure 3. Free harmonic oscillation of the uncoupled pendulums

Table I
Fitted Parameters of the Uncoupled Pendulums

	"Yellow" pendulum	"Green" pendulum
A	25.51 ± .03 cm	24.03 ± .03 cm
ω	2.63 ± .03 rad/s	2.67 ± .03 rad/s

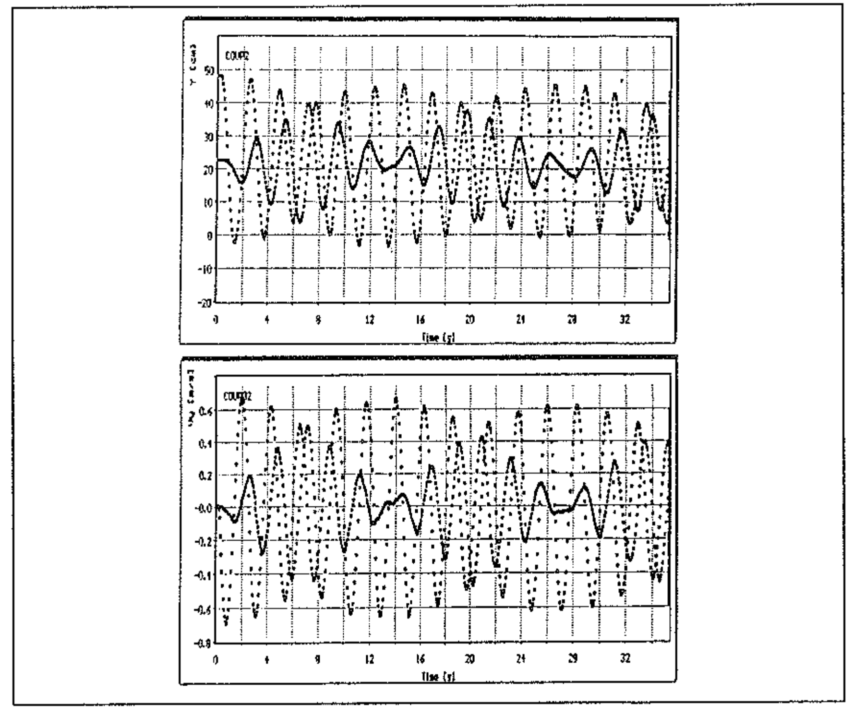

Figure 4. Displacement and velocity-time graphs of two equal-length pendulums (one of them deflected by a small angle at t=0)

COUPLED OSCILLATIONS

We first measured the energy exchange between two equal-length pendulums coupled by a rubber band. The "green" pendulum was deflected by a small angle and the "yellow" one stood still at its equilibrium position. Figure 4 shows the displacement and the velocity of the two pendulums exchanging most of their energy while slightly decaying. The equations for each of the pendulums, fitted by the SAS Nlin procedure (SSUG, 1990), are

Green: $y(t) = 25.91*\cos(.117t)*\cos(2.75t)$
Yellow: $y(t) = 23.54*\sin(.114t)*\cos(2.75t)$

Then, we deflected both pendulums in opposite y-directions and measured their displacements and velocities (see Figure 5), obtaining the following fitted equations:

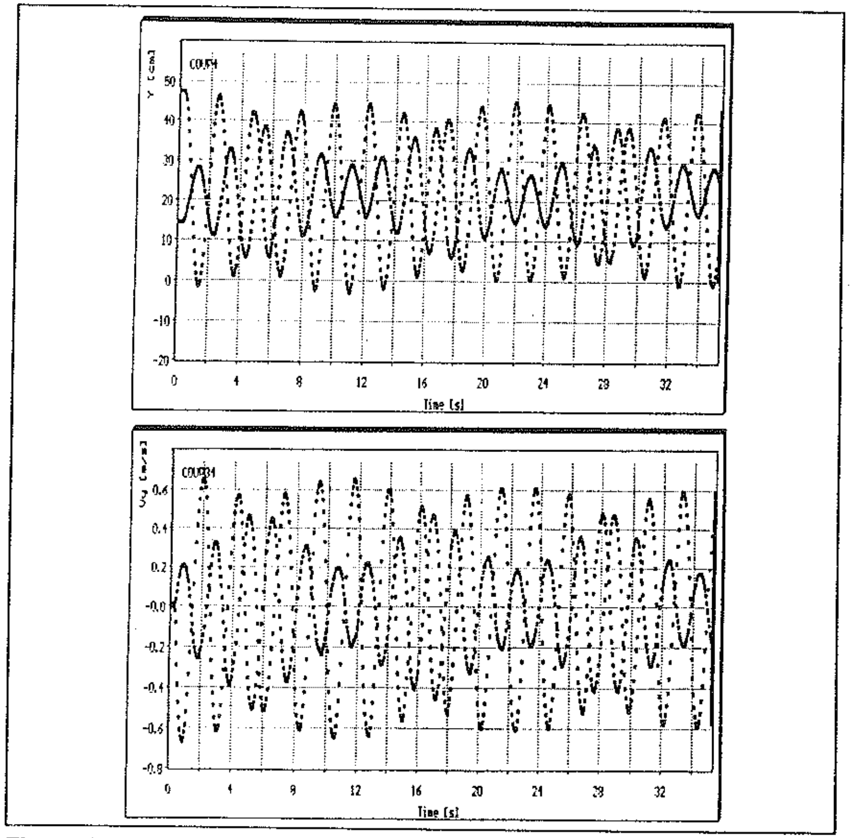

Figure 5. Displacement and velocity-time graphs of two equal-length pendulums (deflected in opposite directions at t=0)

Green: $y(t) = 24.65*\cos(.135t)*\cos(2.83t)$
Yellow: $y(t) = 24.85*\sin(.137t)*\cos(2.83t)$

Although the two amplitudes are not exactly equal, we can clearly see, in both cases, the ongoing energy exchange process between the two pendulums; we can measure its frequency and the frequency of each of the pendulums which is also affected by the coupling. From the fitted equations we can conclude that the coupling constant is greater when the pendulums are initially deflected in opposite directions. We can get a rough approximation of the coupling constant by comparing the experimental equations with Equation (8) and we get $= \delta(.465 \pm .007)$ m/s^2 for one pendulum initially deflected and $\delta = (1.086 \pm .016)$ m/s^2 for both pendulums deflected in opposite directions.

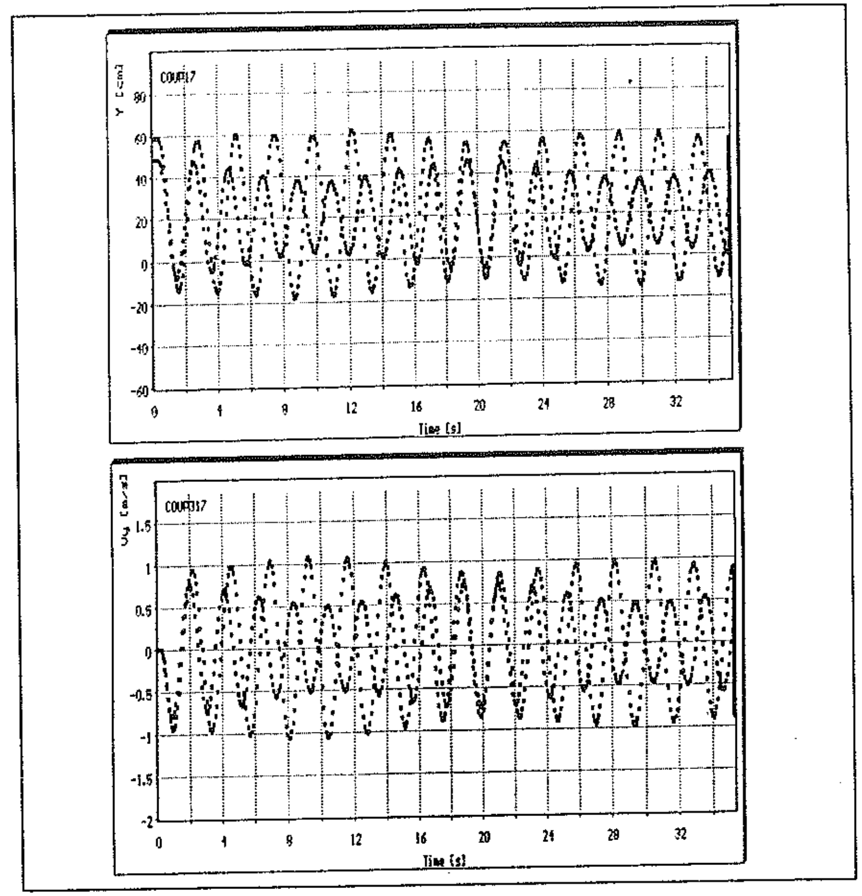

Figure 6. Displacement and velocity-time graphs of two different-length pendulums (deflected in the same direction at t=0)

The third measurement included two coupled pendulums of different lengths (Green-L_1=1.41 m, Yellow-L_2=1.14 m), initially deflected in the same direction (see Figure 6).In this case, we have two pendulums with different frequencies (ω_1=2.635 ± .005 rad/s, ω_2= 2.932 ± .005 rad/s) so that we could foresee a modulated amplitude oscillation of the two coupled pendulums (Alonso & Finn, 1969). The frequency of the amplitude oscillation is expected to be the difference of the frequencies between the two interfering motions, that is (.30 ± .01) rad/s. This is rather well confirmed by the following fitted equations:

Green: y(t)= sqrt{1350+80*cos(.27t)}cos(2.65t)
Yellow: y(t)= sqrt{510+300*cos(.31t)}cos(2.97t)

CHANGING THE HEIGHT OF THE COUPLING POINT

In all the measurements carried out up to now, the rubber band was attached to the pendulums at the same height, 50 cm above the center of mass of the balls. We expect the height of this coupling point (h) to influence the frequency of the pendulums (ω_p) and the frequency of energy exchange (ω_e), because at different coupling heights the force exerted on each of the masses will change.

For instance, Figure 7 shows the displacement of both pendulums for h=1.1 m. We see a very clear energy exchange process leading to the following equations fitted by the Nlin Procedure (SSUG, 1990)

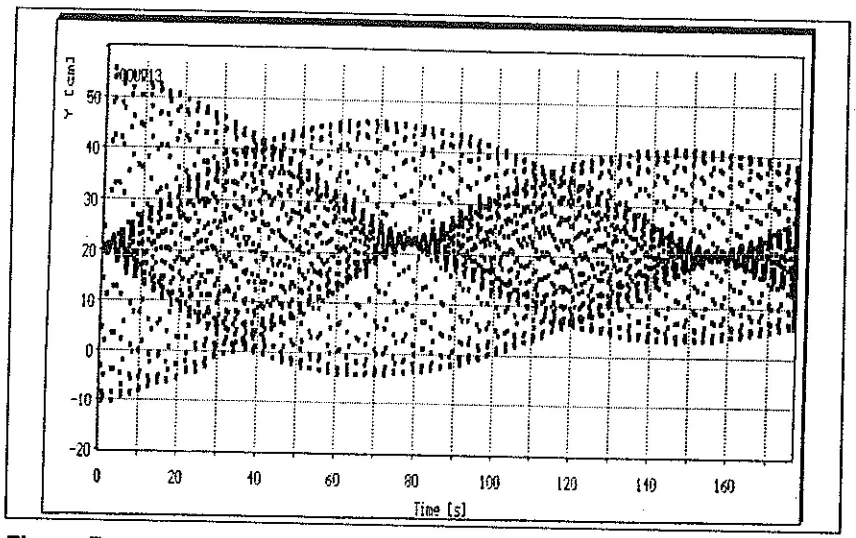

Figure 7. Displacement-time graph of two pendulums coupled at a height of 1.1 m above the center of mass of the balls

Green: y(t) = 33.76*cos(.02t)*cos(2.65t)
Yellow: y(t) = 35.90*sin(.02t)*cos(2.65t)

We made a series of measurements with equal-length pendulums, which are summarized in Table 2.

Table 2
Pendulum Frequency and Energy Exchange Frequency Between Pendulums, as a Function of Coupling Height

Coupling height h(m) ± .01	Pendulum frequency ω_p(rad/s) ± .03	Energy exchange frequency ω_e(rad/s) ± .002
0.2	2.79	.168
0.3	2.77	.149
0.4	2.75	.133
0.5	2.73	.103
0.6	2.71	.085
0.7	2.69	.066
0.8	2.68	.054
0.9	2.67	.039
1.1	2.65	.028

The functional dependence of the pendulum frequency on the coupling height (see Figure 8) may be fitted using some electronic spreadsheet regression method, by the following formula:

$$\omega_p = 2.714/(h+.419)^{0.06}$$

If we put h=1.41 m (pendulums height) in this equation, we get ω_p = 2.617 rad/s, which is within the error limits of the frequency of the "green" uncoupled pendulum.

The functional dependence of the frequency of energy exchange on the coupling height (see Figure 8) may be fitted in the same way by the following formula:

$$\omega_e = 0.286*e^{-1.241h} - .0497$$

As the coupling height increases, the coupling force exerted on the masses decreases and the frequency of energy exchange decreases: It takes a longer time for the energy to be transmitted from one pendulum to the other. If we put h=1.41 m (pendulums height) in this equation, we get we 0 and we have no energy coupling, as it could be theoretically expected.

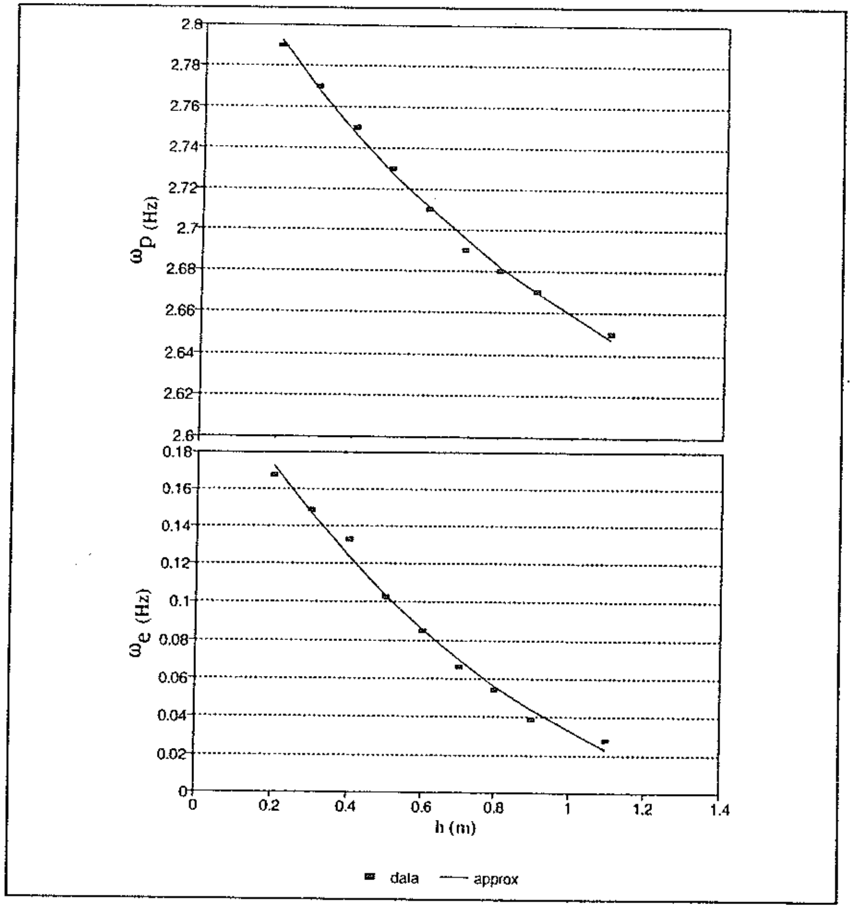

Figure 8. Functional dependence of the pendulum frequency and the energy change frequency on the coupling height

CHANGING THE COUPLING SPRING CONSTANT

In all the measurements carried out up to now, the pendulums were coupled by a rubber-band which was stretched and loosened during the oscillations. In order to investigate the functional dependence of the pendulum frequency (ω_p) and of the energy exchange frequency (ω_e) on the coupling spring constant (k) we replaced the rubber-band with a series of springs. We

made a preliminary static measurement of the different spring constants and then we attached them to a fixed coupling height on both pendulums. For instance, we can see in Figure 9 the displacement of both pendulums for k=0.075 N/m, which shows a very clear energy exchange process leading to the following equations fitted by the Nlin Procedure (SSUG, 1990)

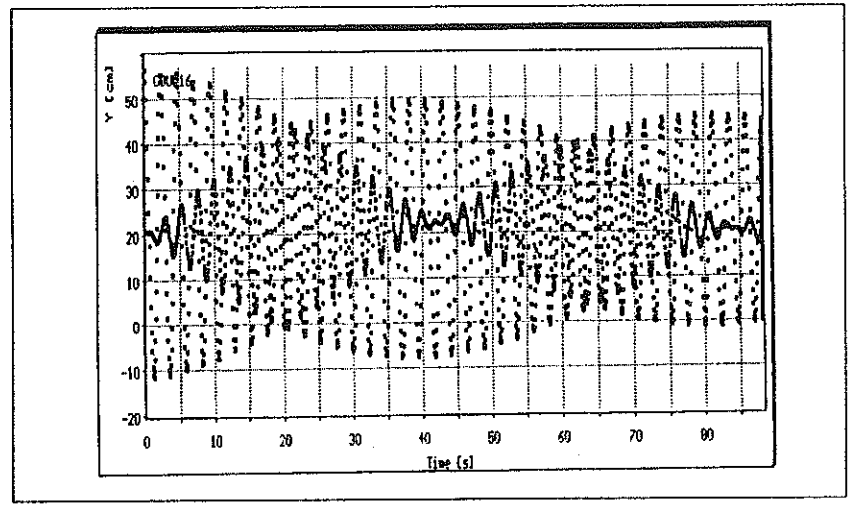

Figure 9. Displacement-time graph of two pendulums coupled by a spring of constant k=0.075 N/m

Green: $y(t) = 30.30 * \cos(.136t) * \cos(2.77t)$
Yellow: $y(t) = 35.16 * \sin(.136t) * \cos(2.75t)$

We made a series of measurements with equal-length pendulums, which are summarized in Table 3.

Table 3

Pendulum Frequency and Energy Exchange Frequency Between Pendulums, as a Function of Spring Constant

Spring constant k (N/m)	Pendulum frequency ω_p(rad/s) ± .03	Energy exchange frequency ω_e(rad/s) ± .002
.075 ± .007	2.76	.037
.15 ± .02	2.70	.075
.30 ± .03	2.67	.136

The functional dependence of the pendulum frequency on the coupling spring constant may be fitted by a linear graph of wp as a function of 1/k (see Figure 10), leading to the formula:

$$\omega_p = 2.64 + 0.009/k$$

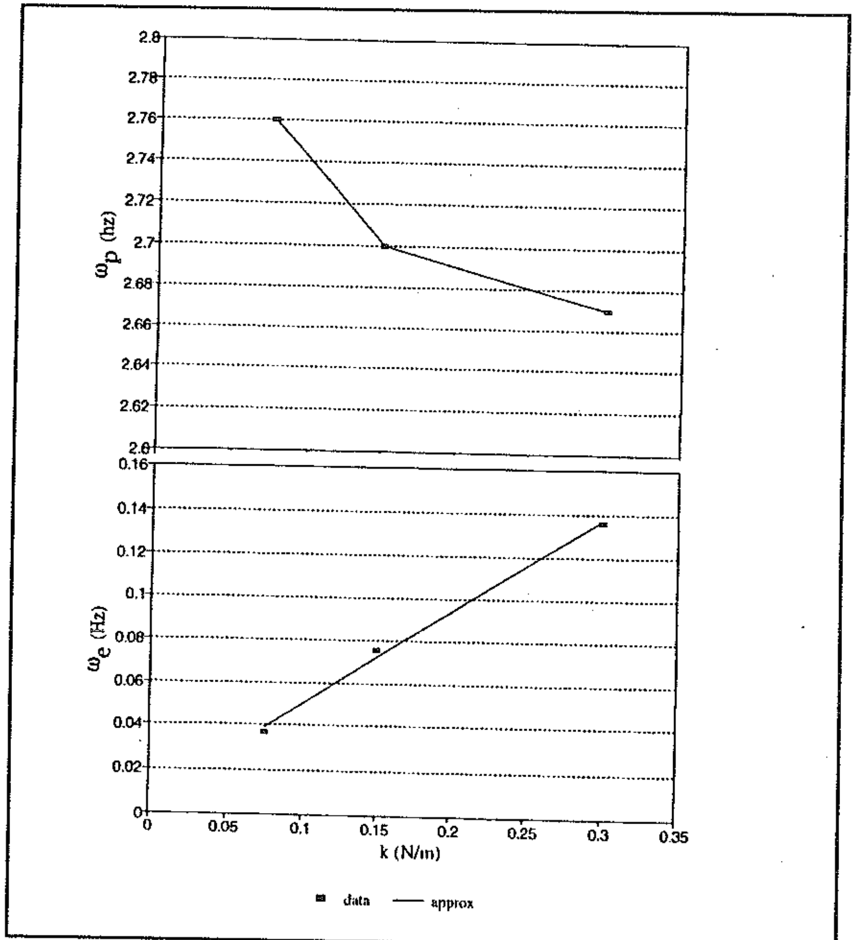

Figure 10. Functional dependence of the pendulum frequency and the energy exchange frequency on the spring constant

The functional dependence of the energy exchange frequency on the coupling spring constant may be fitted by a linear graph (see Figure 10) leading to the formula:

$$\omega_e = 0.007 + 0.435 * k$$

In a totally rigid coupling ($k \to \infty$), the coupled pendulums would oscillate as if they were a single body, with their original frequency, and with zero time for the energy to be transmitted from one pendulum to the other.

DIDACTICAL IMPLEMENTATION AND CONCLUSIONS

The use of V-Scope allows students not only to observe the mechanical oscillations and their graphical representations simultaneously, but also to manipulate a series of a-posteriori analysis tools. Students may save data in a file for future replay of the experiment, using all permissible views and display options for data analysis by another program, or for printing the output of their experiment. Teachers may also use V-Scope in a lecture hall for demonstration purposes, being able to display the ongoing graphs on large screens.

For example, it is possible to move two cursors on a selected graph to measure the time period of the pendulums or their decay, while the cursors' positions and their relative spacing are displayed numerically at the top of the screen.

The experiment may also be replayed while measuring, on line, the positions and frequencies of the pendulums at any given time (see Figure 11), or change the original position-time graph for some spatial graph (Z-Y, for instance) which enables students to display velocity and acceleration vectors of the pendulums, together with their corresponding magnitudes.

The V-Scope was presented in Israeli schools at the beginning of the 1990-91 academic year and until now it has been introduced in more than 100 Israeli schools. Thirty-two teachers who have been interviewed after using V-Scope in their classes, and were informally and generally asked about the profits of using V-Scope in the physics laboratory, emphasized the following characteristics which are important to student learning:

1. Instead of the time-consuming hard work commonly associated with data collection and display in the physics laboratory, students spend more time observing physical phenomena and analyzing abstract representations (graphs) of these phenomena.
2. The data are plotted in graphical ways in real time, so that students get immediate feedback and see the data in a comprehensible form. The immediate feedback helps to make the abstract more concrete.
3. Students learn concepts by investigating the physical phenomena rather than manipulating symbols as in traditional courses. They spend a large portion of their laboratory time observing, interpreting, and discussing data with their peers.

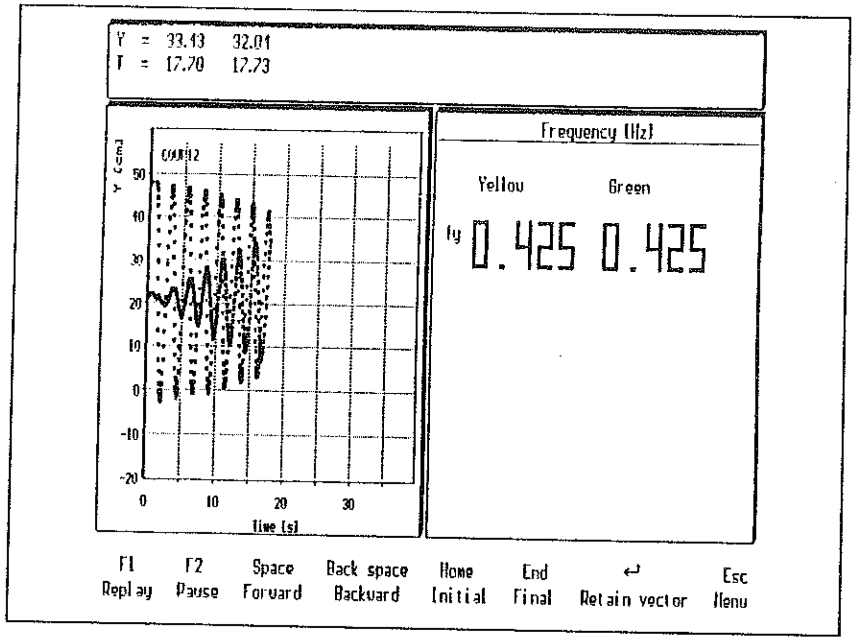

Figure 11. Measuring the positions and frequencies of the pendulums

In senior high school, Israeli students select a major field of study on which they are evaluated by a matriculation examination at the end of the 12th grade. In the last two years, many V-Scope controlled laboratory experiments (e.g., Newton's Second Law, the momentum conservation principle, the oscillatory motion of a mass attached to a spring) were included in the physics matriculation examination. We interviewed many of the students who participated in such examinations, and they emphasized especially the user friendliness of the software and the hardware and the possibility to concentrate on the physical phenomena instead of on the time consuming and tedious work of data collection.

We believe that "the time students now spend passively listening to lectures" (Laws, 1991, p. 26) will be better spent by conducting direct inquiries like those described in this study.

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Notes

1. Litek Advanced Systems, Ltd. 14 Eser-Tahanot St. (P.O. Box 13261), Tel Aviv, Israel.
2. PASCO Scientific, 10101 Foothills Blvd., P.O. Box 619011, Roseville, CA, USA.